

# A Comment on "Semiquantum Chaos"

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## Abstract

The identification of the particle creation and distruction operators is discussed.

In a letter<sup>1</sup> Cooper et al. (henceforth CDMS) considered a system, in which a classical oscillator interacts with a purely quantum mechanical oscillator, described by the classical Lagrangian:

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{A}^2 - \frac{1}{2}(m^2 + e^2 A^2)x^2 \quad (1)$$

The scope of this comment is to correctly identify the particle creation and destruction operators. The corresponding time dependent occupation number (differing from eq. (20) of CDMS) leads to changes in analytical and numerical results. Our results are obtained through the use of the Born-Oppenheimer approach (which is pertinent in the presence of two mass, or time, scales) and the method of adiabatic invariants<sup>2</sup>.

From (1) one obtains for a state of energy  $E$  a Schrödinger equation:

$$\frac{1}{2}\left(-\hbar^2 \frac{\partial^2}{\partial x^2} + \omega^2 x^2 - \hbar^2 \frac{\partial^2}{\partial A^2} - 2E\right)\Psi_E(x, A) = 0 \quad (2)$$

where  $\omega^2 = m^2 + e^2 A^2$ . On factorizing  $\Psi_E(x, A) = \psi(A)\chi(x, A)$  (we omit the index  $E$  for simplicity) one obtains the coupled equations:

$$\left[-\frac{\hbar^2}{2}D^2 - E + \langle \hat{H}_x \rangle\right]\psi = \frac{\hbar^2}{2}\langle \bar{D}^2 \rangle\psi \quad (3)$$

$$(\hat{H}_x - \langle \hat{H}_x \rangle)\chi - \frac{\hbar^2}{\psi}(D\psi)\bar{D}\chi = \frac{\hbar^2}{2}(\bar{D}^2 - \langle \bar{D}^2 \rangle)\chi \quad (4)$$

where (4) exists where  $\psi$  has support; for an operator  $\hat{O}$ ,  $\langle \hat{O} \rangle = \int dx \chi^* \hat{O} \chi / \int dx \chi^* \chi$ ,  $D = \frac{\partial}{\partial A} + \langle \frac{\partial}{\partial A} \rangle$ ,  $\bar{D} = \frac{\partial}{\partial A} - \langle \frac{\partial}{\partial A} \rangle$  and  $\hat{H}_x$  is given by the first two terms on the LHS of (2). On considering the semiclassical approximation to  $\psi \simeq \frac{1}{\sqrt{A}} \exp(-\int^A \langle \frac{\partial}{\partial A'} \rangle dA' + \frac{i}{\hbar} \int^A \dot{A}' dA')$  and neglecting the RHS of (3) (fluctuations), one reproduces the Hamilton-Jacobi equation for  $A$  (eq. (13) of CDMS) and consequently the time evolution equations. The same semiclassical approximation for  $\psi$  and the neglect of the RHS in eq. (4) leads to the Schrödinger equation:

$$(\hat{H}_x - i\hbar \frac{\partial}{\partial t})\chi_s = 0 \quad (5)$$

where  $\chi_s = \exp[-\int^t dt' (\frac{i}{\hbar} \langle \hat{H}_x \rangle + \langle \frac{\partial}{\partial A'} \rangle \dot{A}')] \chi$ .  $\hat{H}_x$  can be factorized as:

$$\hat{H}_x = \hbar\omega(b^\dagger b + \frac{1}{2}) \quad (6)$$

where  $b = \sqrt{\frac{\omega}{2\hbar}}(x + \frac{\hbar}{\omega} \frac{\partial}{\partial x})$  thus allowing us to identify the charged particle creation and destruction operators.

In order to obtain solutions to the Schrödinger equation it is convenient to introduce the Hermitian adiabatic invariant (satisfying  $\frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar} [\hat{I}, \hat{H}] = 0$ ):

$$\hat{I} = \hbar(a^\dagger a + \frac{1}{2}) \quad (7)$$

where  $a = e^{i\theta} (\frac{\Omega}{2\hbar})^{\frac{1}{2}} [x(1 + \frac{i}{2} \frac{\dot{\Omega}}{\Omega^2}) + \frac{\hbar}{\Omega} \frac{\partial}{\partial x}]$  is a linear (non-hermitian) adiabatic invariant, which corresponds to the same operator used by CDMS in the Schrödinger representation, and  $\theta = \int^t dt' \Omega$ . Further  $\Omega$  satisfies

$$\frac{1}{2} \ddot{\Omega} - \frac{3}{4} (\frac{\dot{\Omega}}{\Omega})^2 + \Omega^2 = \omega^2 \quad (8)$$

On solving eq. (8) one knows the evolution of the quantum system. The  $a$  and  $b$  operators are related by a Bogolubov transformation: however while our  $a$ 's agree with those of CDMS, the  $b$ 's do not. Indeed one sees that with their choice the RHS of (6) corresponds to

$$-\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + \frac{x^2}{2} [\omega^2 + \frac{1}{4} (\frac{\dot{\omega}}{\omega})^2] - i\hbar \frac{\dot{\omega}}{4\omega} \{x, \frac{\partial}{\partial x}\} \quad (9)$$

On defining the vacuum state by  $a|0\rangle = 0$  we can compute the following average quantities for  $|\chi\rangle = |0\rangle$ :

$$\langle x^2 \rangle = \frac{\hbar}{2\Omega} \quad ; \quad \langle \dot{x}^2 \rangle = \frac{\hbar}{2} \left( \Omega + \frac{1}{4} \frac{\dot{\Omega}^2}{\Omega^3} \right) \quad (10)$$

Finally the average time dependent particle number is:

$$\langle b^\dagger b \rangle = \frac{1}{4} \left( \frac{\omega}{\Omega} + \frac{\Omega}{\omega} + \frac{1}{4} \frac{\dot{\Omega}^2}{\omega \Omega^3} \right) - \frac{1}{2} \quad (11)$$

It is straightforward to verify that the expressions in (10) agree with the corresponding ones of CDMS while (11) does not. In order to better quantify this we have approximately evaluated eq. (11) in the two cases (ours and ref<sup>1</sup>) for  $e^2 A^2 / m^2 < 1$  and find that they differ by a term:

$$\frac{e^4}{16m^6} \dot{A}^2 A^2 + O(e^6) \quad (12)$$

which, depending on  $A$ , can be significant.

We have also seen, by a numerical analysis similar to the one of CDMS, that although the general time behaviour of  $\langle b^\dagger b \rangle$  is similar, our expression (11) exhibits more structure (see figure). Let us end by noting that one may also estimate the neglected terms, finding that both the RHS of eq. (3) and (4) and the non-leading term arising from the prefactor in the semiclassical limit for  $\psi$  give corrections of the order  $\frac{\hbar^2 e^2}{m^2} \langle b^\dagger b \rangle$ .

## REFERENCES

1. F. Cooper, J. F. Dawson, D. Meredith and H. Shepard, Phys. Rev. Lett. **72**, 1337 (1994)  
and references therein.
2. H. R. Lewis and B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969)

